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# Hermite-Hadamard Inequalities Type Using Fractional Integrals for MT-convex Stochastic Process

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## Abstract

By applying the standard fractional integral operator of Riemann-Liouville on MT-convex stochastic processes, we can obtain new inequalities of Hermite-Hadamard, providing in the process new estimates on these types of Hermite-Hadamard inequalities for stochastic process whose first derivatives absolute values are MT-convex.

**Keywords:** Hermite-Hadamard inequality; fractional integral; MT-convex Stochastic process.

# 1 Introduction and Preliminary

## 1.1 Preliminary

The first investigator of the study of stochastic convex processes was Nagy 1974 ([13]), using their measurable aspect to find solutions to the additive generalization of Cauchy functional equation. Both Nikodem in 1980 (see [14]) and later Skowronski in 1995 (see [21]) were able to generalize the properties of convex functions to stochastic process, in order to obtain several new results, among which we mention those found by Kotrys for convex stochastic processes in 2012 (see [7]), and the ones found by E. Tomar concerning second sense stochastic processes in 2014 (see [19]), more details on this subject can be found in the following references [2, 15] and [20].

Moreover and almost in all areas of mathematics, inequalities can have an important role, mostly because of their applications that can be found in various fields of science such as, physical sciences and engineering sciences. The Hermite-Hadamard inequation is not excluded from these facts, especially due to its rich geometrical significance and various applications in [17, 9] and [18], making it very significant for convex functions, and thus an interesting object of a study. Furthermore, the rich and diverse applications of the Hermite-Hadamard inequality highlight its continuing importance as a fundamental result in convex analysis, and suggest that it will continue to play an important role in future research across a wide range of mathematical and scientific fields, as well as a source of inspiration for finding similar results for other types of inequality in combination with fractional calculus [6], such as Ostrowski inequalities type [10] and provide a new approach to solve fractional differential systems and equations [16].

Consequently in recent years, researchers have explored the possibility to extend the results of the Hermite-Hadamard inequality to stochastic processes using fractional calculus, in order to give some estimation of the error in the approximation of its sides, giving birth to resherchs done on functions using different types of fractional intagral operators like a generalized $(k, s)$ Riemann-Liouville operator in [1], and by also exploring different types of convexities in [5] and [22].

In the same spirit the aim of this work is the application of the Liouville-Riemann fractional integral operator in order to establish some Hermite-Hadamard type inequalities for MT-convex stochastic processes allowing us new estimations on these types of inequalities for stochastic processes whose first derivatives absolute values are MT-convex.

## 1.2 Definitions

Considering the probability space $(\mathcal{E}, \mathcal{T}, \mathrm{P})$. A function $\Re_v : \mathcal{E} \to \mathbb{R}$ is said to be a random variable if $\mathcal{T}$ is measurable. In an interval $\mathcal{I} \subset \mathbb{R}$, a function $\mathcal{S}_p : \mathcal{I} \times \mathcal{E} \to \mathbb{R}$ is considered a stochastic process if $\mathcal{S}_p(\chi, \cdot)$ is a random variable $\forall \chi \in \mathcal{I}$.

**Definition 1.1.** (*see* [11, 20])
*$\mathcal{S}_p : \mathcal{I} \times \mathcal{E} \to \mathbb{R}$ is known to be:*

1. *Continuous in probability in $\mathcal{I}$, if $\forall \chi_0 \in \mathcal{I}$:*

$$\mathrm{P} - \lim_{\chi \to \chi_0} \mathcal{S}_p(\chi, \cdot) = \mathcal{S}_p(\chi_0, \cdot), \tag{1}$$

$\mathrm{P}-\lim$ *being the limit in probability, (see [11, 8]).*

2. *Mean square continuous (MS-C) in $\mathcal{I}$, if $\forall \chi_0 \in \mathcal{I}$:*

$$\lim_{\chi \to \chi_0} \mathrm{E}\left[\left(\mathcal{S}_p(\chi,\cdot) - \mathcal{S}_p(\chi_0,\cdot)\right)^2\right] = 0, \tag{2}$$

*where $\mathrm{E}\left[\mathcal{S}_p(x,\cdot)\right]$ is the expectation value of the random variable $\mathcal{S}_p(\chi,\cdot)$.*

3. *Mean-square differentiable (MS-D) at a point $\chi \in \mathcal{I}$, if a random variable $\mathcal{S}'_p(\chi,\cdot) : \mathcal{E} \to \mathbb{R}$ exists verifying:*

$$\forall \chi_0 \in \mathcal{I} : \lim_{\chi \to \chi_0} E\left[\frac{\mathcal{S}_p(\chi,\cdot) - \mathcal{S}_p(\chi_0,\cdot)}{\chi - \chi_0} - \mathcal{S}'_p(\chi,\cdot)\right]^2 = 0. \tag{3}$$

**Definition 1.2.** (*see* [15, 11, 8])
*Considering a stochastic process $\mathcal{S}_p : \mathcal{I} \times \mathcal{E} \to \mathbb{R}$ with $\mathrm{E}\left[\mathcal{S}_p(\chi,\cdot)^2\right] \prec \infty$, and $\mu = \chi_0 \prec \chi_1 \prec \chi_2 \prec \chi_3 \prec \ldots\ldots \prec \chi_n = \nu$ a partition of $[\mu;\nu] \subset \mathcal{I}$, $\varsigma_k \in \left[\chi_{k-1};\chi_k\right]$ for $k = 1,\ldots,n$.*

*For random variable $\Re_v : \mathcal{E} \to \mathbb{R}$ to be mean-square integral (MS-I) of the stochastic process $\mathcal{S}_p$ on $[\mu;\nu]$, the following equality must be satisfied:*

$$\lim_{n \to \infty} \mathrm{E}\left[\left(\sum_{k=1}^{k=n} \mathcal{S}_p(\varsigma_k,\cdot)(\tau_k - \tau_{k-1}) - \Re_v(\cdot)\right)^2\right] = 0, \tag{4}$$

*for every normal sequence of partitions of $[\mu;\nu]$ and $\forall \varsigma_k \in [\chi_{k-1};\chi_k]$ for $k = 1,\ldots,n$. Thus we can write:*

$$\Re_v(\cdot) = \int_\mu^\nu \mathcal{S}_p(\chi,\cdot)d\chi. \tag{5}$$

**Definition 1.3.** *$\mathcal{S}_p : \mathcal{I} \times \mathcal{E} \to \mathbb{R}$ is $MT$-convex stochastic process, if $\forall \chi \in [\mu;\nu], \mathcal{S}_p(\chi,) \geq 0$ with $\mu;\nu \in \mathcal{I} \subset \mathbb{R}$, and $\forall \xi \in [0,1]$, the inequality:*

$$\mathcal{S}_p\Big(\xi\mu + (1-\xi)\nu,\cdot\Big) \leq \frac{\sqrt{\xi}}{2\sqrt{1-\xi}}\mathcal{S}_p(\mu,\cdot) + \frac{\sqrt{1-\xi}}{2\sqrt{\xi}}\mathcal{S}_p(\nu,\cdot), \tag{6}$$

*is satisfied.*

**Theorem 1.1.** (*Hermite-Hadamard inequality for Jensen convex stochastic process*) (*see* [7])
*If $\mathcal{S}_p : \mathcal{I} \times \mathcal{E} \to \mathbb{R}$ is Jensen-convex as well as being MS-C on $\mathcal{I}$ stochastic process, then $\forall(\mu,\nu) \in \mathcal{I}^2$, we have:*

$$\mathcal{S}_p\left(\frac{\mu+\nu}{2},\cdot\right) \leq \frac{1}{\nu-\mu}\int_\mu^\nu \mathcal{S}_p(\chi,\cdot)d\chi \leq \frac{\mathcal{S}_p(\mu,\cdot) + \mathcal{S}_p(\nu,\cdot)}{2}. \tag{7}$$

**Definition 1.4.** (*see* [4, 12, 3])
*$\mathcal{S}_p$ is MS-I on $[\mu,\nu]$. The Riemann-Liouville integrals $I^\alpha_{\mu^+}(\mathcal{S}_p)$ and $I^\alpha_{\nu^-}(\mathcal{S}_p)$ of order $\alpha > 0$, with $\mu \geq 0$ are defined by:*

$$I^\alpha_{\mu+}\mathcal{S}_p(\chi) = \frac{1}{\xi(\alpha)}\int_\mu^\chi (\chi-\tau)^{\alpha-1}\mathcal{S}_p(\tau,\cdot)d\tau, \qquad \chi > \mu, \tag{8}$$

*and:*

$$I^\alpha_{\nu^-}\mathcal{S}_p(\chi) = \frac{1}{\xi(\alpha)}\int_\chi^\nu (\tau-\chi)^{\alpha-1}\mathcal{S}_p(t,\cdot)d\tau, \qquad \chi < \nu, \tag{9}$$

*with $\xi(\alpha) = \int_0^\infty e^{-s}u^{\alpha-1}\,\mathrm{d}s$.*
*For $\alpha = 1$, we return to the classical integral.*

## 2 Mean Results

The following Lemma is essential to properly prove the following results.

**Lemma 2.1.** *Let $\mathcal{S}_p : \mathcal{I} \times \mathcal{T} \to \mathbb{R}$ be a MS-D stochastic process on $\mathcal{I}^\circ, \mu, \nu \in \mathcal{I}^\circ$ with $\mu < \nu$. If $\mathcal{S}'_p$ is MS-I on $[\mu, \nu]$ then the following equality holds:*

$$\frac{(\nu-\chi)\mathcal{S}_p(\nu,\cdot)+(\chi-\mu)\mathcal{S}_p(\mu,\cdot)}{(\nu-\mu)} - \frac{1}{(\nu-\mu)}\int_\mu^\nu \mathcal{S}_p(u,\cdot)du$$
$$= \frac{(\chi-\mu)^2}{(\nu-\mu)}\int_0^1 (\tau-1)\mathcal{S}'_p\Big(\chi\tau+(1-\tau)\mu,\cdot\Big)d\tau + \frac{(\nu-\chi)^2}{(\nu-\mu)}\int_0^1 (1-\tau)\mathcal{S}'_p\Big(\chi\tau+(1-\tau)\nu,\cdot\Big)d\tau,$$

*for each $\chi \in [\mu, \nu]$.*

*Proof.* Considering :

$$\mathcal{F} = \frac{(\chi-\mu)^2}{(\nu-\mu)}\int_0^1 (\tau-1)\mathcal{S}'_p\Big(\chi\tau+(1-\tau)\mu,\cdot\Big)\,d\tau + \frac{(\nu-\chi)^2}{(\nu-\mu)}\int_0^1 (1-\tau)\mathcal{S}'_p\Big(\chi\tau+(1-\tau)\nu,\cdot\Big)\,d\tau,$$

by calculating the intergals we obtain:

$$\begin{aligned}\mathcal{F} &= \frac{(\chi-\mu)^2}{\nu-\mu}\left[(\tau-1)\frac{\mathcal{S}_p\Big(\tau\chi+(1-\tau)\mu,\cdot\Big)}{\chi-\mu}\Bigg|_0^1 - \int_0^1 \frac{\mathcal{S}_p\Big(\tau\chi+(1-\tau)\mu,\cdot\Big)}{\chi-\mu}d\tau\right] \\ &\quad + \frac{(\nu-\chi)^2}{\nu-\mu}\left[(1-\tau)\frac{\mathcal{S}_p\Big(\tau\chi+(1-\tau)\nu,\cdot\Big)}{\chi-\nu}\Bigg|_0^1 + \int_0^1 \frac{\mathcal{S}_p\Big(\tau\chi+(1-\tau)\nu,\cdot\Big)}{\chi-\nu}d\tau\right] \\ &= \frac{(\nu-\chi)\mathcal{S}_p(\nu,\cdot)+(\chi-\mu)\mathcal{S}_p(\mu,\cdot)}{(\nu-\mu)} - \frac{1}{(\nu-\mu)}\int_\mu^\nu \mathcal{S}_p(u,\cdot)du.\end{aligned}$$

□

We propose following refinement of the Hermite-Hadamard type inequality.

**Theorem 2.1.** *Let $\mathcal{S}_p : \mathcal{I} \times \mathcal{T} \to \mathbb{R}$ be a MS-D stochastic process on $\mathcal{I}^\circ, \mu, \nu \in \mathcal{I}^\circ$ with $\mu < \nu$. If $\mathcal{S}'_p$ is MS-I on $[\mu, \nu]$, and $\left|\mathcal{S}'_p\right|$ is MT-convex on $[\mu, \nu]$, with $|\mathcal{S}'_p(\chi, \cdot)| \le A, \chi \in [\mu, \nu]$, then we have:*

$$\left|\frac{(\nu-\chi)\mathcal{S}_p(\nu,\cdot)+(\chi-\mu)\mathcal{S}_p(\mu,\cdot)}{(\nu-\mu)} - \frac{1}{(\nu-\mu)}\int_\mu^\nu \mathcal{S}_p(u,\cdot)du\right| \le \frac{A\pi\Big[(\chi-\mu)^2+(\nu-\chi)^2\Big]}{4(\nu-\mu)},$$

*for each $\chi \in [\mu, \nu]$.*

*Proof.* With the help of Lemma 2.1 and MT-covexity on $|\mathcal{S}'_p|$, it follows that:

$$\begin{aligned}&\left|\frac{(\nu-\chi)\mathcal{S}_p(\nu,\cdot)+(\chi-\mu)\mathcal{S}_p(\mu,\cdot)}{(\nu-\mu)}-\frac{1}{(\nu-\mu)}\int_\mu^\nu \mathcal{S}_p(u,\cdot)du\right|\\ &\leq \frac{(\chi-\mu)^2}{\nu-\mu}\int_0^1(1-\tau)\left[\frac{\sqrt{\tau}}{2\sqrt{1-\tau}}\left|\mathcal{S}'_p(\chi,\cdot)\right|+\frac{\sqrt{1-\tau}}{2\sqrt{\tau}}\left|\mathcal{S}'_p(\mu,\cdot)\right|\right]d\tau\\ &\quad+\frac{(\nu-\chi)^2}{\nu-\mu}\int_0^1(1-\tau)\left[\frac{\sqrt{\tau}}{2\sqrt{1-\tau}}\left|\mathcal{S}'_p(\chi,\cdot)\right|+\frac{\sqrt{1-\tau}}{2\sqrt{\tau}}\left|\mathcal{S}'_p(\nu,\cdot)\right|\right]d\tau\\ &\leq \frac{A\left[(\chi-\mu)^2+(\nu-\chi)^2\right]}{2(\nu-\mu)}\int_0^1\left(\tau^{1/2}(1-\tau)^{1/2}+\tau^{-1/2}(1-\tau)^{3/2}\right)d\tau,\end{aligned}$$

using Euler Beta function defined by : $\beta(\mu,\nu)=\int_0^1\tau^{\mu-1}(1-\tau)^{\nu-1}d\tau$, where $\mu,\nu>0$, we get the result. □

**Example 2.1.** *If* $\chi=\dfrac{(\mu+\nu)}{2}$ *in Theorem 2.1, we obtain:*

$$\left|\frac{\mathcal{S}_p(\mu,\cdot)+\mathcal{S}_p(\nu,\cdot)}{2}-\frac{1}{(\nu-\mu)}\int_\mu^\nu \mathcal{S}_p(u,\cdot)du\right|\leq\frac{A\pi(\nu-\mu)}{8}.$$

By incorporating the power of the absolute value of the first derivative, using Hölder's inequality, we can obtain a new Hermite-Hadamard inequality for MT-convex stochastic processes.

**Theorem 2.2.** *Let* $\mathcal{S}_p:\mathcal{I}\times\mathcal{T}\to\mathbb{R}$ *be a MS-D stochastic process on* $\mathcal{I}^\circ$, $\mu,\nu\in\mathcal{I}^\circ$ *with* $\mu<\nu$. *If* $\mathcal{S}'_p$ *is MS-I on* $[\mu,\nu]$ $\left|\mathcal{S}'_p\right|^\sigma$ *is MT-convex on* $[\mu,\nu]$, *for some* $\sigma>1$ *with* $|\mathcal{S}'_p(\chi,\cdot)|\leq A,\chi\in[\mu,\nu]$, *then we have:*

$$\left|\frac{(\nu-\chi)\mathcal{S}_p(\nu,\cdot)+(\chi-\mu)\mathcal{S}_p(\mu,\cdot)}{(\nu-\mu)}-\frac{1}{(\nu-\mu)}\int_\mu^\nu \mathcal{S}_p(u,\cdot)du\right|\leq\frac{A}{(1+\rho)^{1/\rho}}\left(\frac{\pi}{2}\right)^{\frac{1}{\sigma}}\frac{(\chi-\mu)^2+(\nu-\chi)^2}{(\nu-\mu)},$$

*for each* $\chi\in[\mu,\nu]$, *with:* $\dfrac{1}{\rho}+\dfrac{1}{\sigma}=1$.

*Proof.* Suppose that $\rho>1$. With the help of Hölder inequality on Lemma 2.1, we have:

$$\begin{aligned}&\left|\frac{(\nu-\chi)\mathcal{S}_p(\nu,\cdot)+(\chi-\mu)\mathcal{S}_p(\mu,\cdot)}{(\nu-\mu)}-\frac{1}{(\nu-\mu)}\int_\mu^\nu \mathcal{S}_p(u,\cdot)du\right|\\ &\leq\frac{(\chi-\mu)^2}{(\nu-\mu)}\left(\int_0^1(1-\tau)^\rho d\tau\right)^{\frac{1}{\rho}}\left(\int_0^1\left|\mathcal{S}'_p\Big(\chi\tau+(1-\tau)\mu,\cdot\Big)\right|^\sigma d\tau\right)^{\frac{1}{\sigma}}\\ &\quad+\frac{(\nu-\chi)^2}{(\nu-\mu)}\left(\int_0^1(1-\tau)^\rho d\tau\right)^{\frac{1}{\rho}}\left(\int_0^1\left|\mathcal{S}'_p\Big(\chi\tau+(1-\tau)\nu,\cdot\Big)\right|^\sigma d\tau\right)^{\frac{1}{\sigma}},\end{aligned}$$

and by using the MT-convexity of $|\mathcal{S}'_p|^\sigma$ and $|\mathcal{S}'_p(\chi,\cdot)|\leq A$, then we have:

$$\begin{aligned}&\int_0^1\left|\mathcal{S}'_p\Big(\chi\tau+(1-\tau)\mu,\cdot\Big)\right|^\sigma d\tau\leq\int_0^1\left[\frac{\sqrt{\tau}}{2\sqrt{1-\tau}}|\mathcal{S}'_p(\chi,\cdot)|^\sigma+\frac{\sqrt{1-\tau}}{2\sqrt{\tau}}|\mathcal{S}'_p(\mu,\cdot)|^\sigma\right]d\tau\\ &=\frac{\pi}{4}\left[|\mathcal{S}'_p(\chi,\cdot)|^\sigma+|\mathcal{S}'_p(\mu,\cdot)|^\sigma\right]\leq\frac{\pi}{2}A^\sigma,\end{aligned}$$

and similarly:

$$\int_0^1 \left|\mathcal{S}_p'\Big(\chi\tau+(1-\tau)\nu,\cdot\Big)\right|^\sigma d\tau \le \frac{\pi}{4}\Big[|\mathcal{S}_p'(\chi,\cdot)|^\sigma+|\mathcal{S}_p'(\nu,\cdot)|^\sigma\Big] \le \frac{\pi}{2}A^\sigma.$$

Therefore, we have:

$$\left|\frac{(\nu-\chi)\mathcal{S}_p(\nu,\cdot)+(\chi-\mu)\mathcal{S}_p(\mu,\cdot)}{(\nu-\mu)}-\frac{1}{(\nu-\mu)}\int_\mu^\nu \mathcal{S}_p(u,\cdot)du\right| \le \frac{A}{(1+\rho)^{1/\rho}}\left(\frac{\pi}{2}\right)^{\frac{1}{\sigma}}\frac{(\chi-\mu)^2+(\nu-\chi)^2}{(\nu-\mu)}.$$

□

**Example 2.2.** *For* $\chi=\dfrac{(\mu+\nu)}{2}$*, the result in Theorem 2.2 becomes:*

$$\left|\frac{\mathcal{S}_p(\mu,\cdot)+\mathcal{S}_p(\nu,\cdot)}{2}-\frac{1}{(\nu-\mu)}\int_\mu^\nu \mathcal{S}_p(u,\cdot)du\right| \le \frac{A\pi^{\frac{1}{\sigma}}}{(1+\rho)^{1/\rho}}\left(\frac{1}{2}\right)^{1+\frac{1}{\sigma}}(\nu-\mu).$$

**Theorem 2.3.** *Let* $\mathcal{S}_p:\mathcal{I}\times\mathcal{T}\to\mathbb{R}$ *be a MS-D stochastic process on* $I^\circ, a,b\in\mathcal{I}^\circ$ *with* $\mu<\nu$*. If* $\mathcal{S}_p'$ *is MS-I on* $[\mu,\nu]$ $\left|\mathcal{S}_p'\right|^\sigma$ *is MT-convex on* $[\mu,\nu]$*, for some* $\sigma>1$ *with* $|\mathcal{S}_p'(\chi,\cdot)|\le A, \chi\in[\mu,\nu]$*, then we have:*

$$\left|\frac{(\nu-\chi)\mathcal{S}_p(\nu,\cdot)+(\chi-\mu)\mathcal{S}_p(\mu,\cdot)}{(\nu-\mu)}-\frac{1}{(\nu-\mu)}\int_\mu^\nu \mathcal{S}_p(u,\cdot)du\right| \le A\left(\frac{1}{2}\right)^{1+\frac{1}{\sigma}}\pi^{\frac{1}{\sigma}}\frac{(\chi-\mu)^2+(\nu-\chi)^2}{(\nu-\mu)}.$$

*Proof.* With the help of Hölder inequality on Lemma 2.1, we have:

$$\begin{aligned}&\left|\frac{(\nu-\chi)\mathcal{S}_p(\nu,\cdot)+(\chi-\mu)\mathcal{S}_p(\mu,\cdot)}{(\nu-\mu)}-\frac{1}{(\nu-\mu)}\int_\mu^\nu \mathcal{S}_p(u,\cdot)\mathrm{d}u\right|\\ &\le \frac{(\chi-\mu)^2}{(\nu-\mu)}\left(\int_0^1(1-\tau)d\tau\right)^{1-\frac{1}{\sigma}}\left(\int_0^1(1-\tau)\left|\mathcal{S}_p'\Big(\chi\tau+(1-\tau)\mu,\cdot\Big)\right|^\sigma d\tau\right)^{\frac{1}{\sigma}}\\ &\quad+\frac{(\nu-\chi)^2}{(\nu-\mu)}\left(\int_0^1(1-\tau)d\tau\right)^{1-\frac{1}{\sigma}}\left(\int_0^1(1-\tau)\left|\mathcal{S}_p'\Big(\chi\tau+(1-\tau)\nu,\cdot\Big)\right|^\sigma d\tau\right)^{\frac{1}{\sigma}},\end{aligned}$$

and since $|\mathcal{S}_p'|^\sigma$ is MT-convex stochastic processus, with $|\mathcal{S}_p'(\chi,\cdot)|\le A$, $\forall\chi\in[\mu,\nu]$, we get:

$$\begin{aligned}&\int_0^1(1-\tau)\left|\mathcal{S}_p'\Big(\chi\tau+(1-\tau)\mu,\cdot\Big)\right|^\sigma d\tau\\ &\le\frac{1}{2}|\mathcal{S}_p'(\chi,\cdot)|^\sigma\int_0^1\tau^{1/2}(1-\tau)^{1/2}d\tau+\frac{1}{2}|\mathcal{S}_p'(\mu,\cdot)|^\sigma\int_0^1\tau^{-1/2}(1-\tau)^{3/2}d\tau=\frac{\pi}{4}A^\sigma.\end{aligned}$$

□

**Example 2.3.** *If* $\chi=\dfrac{(\mu+\nu)}{2}$*, then the result in Theorem 2.3 becomes:*

$$\left|\frac{\mathcal{S}_p(\mu,\cdot)+\mathcal{S}_p(\nu,\cdot)}{2}-\frac{1}{(\nu-\mu)}\int_\mu^\nu \mathcal{S}_p(u,\cdot)du\right| \le A\pi^{\frac{1}{\sigma}}\left(\frac{1}{2}\right)^{2+\frac{1}{\sigma}}(\nu-\mu).$$

**Theorem 2.4.** *Let $\mathcal{S}_p : \mathcal{I} \times \mathcal{T} \to \mathbb{R}$ be a MS-D stochastic process on $I^\circ, a, b \in \mathcal{I}^\circ$ with $\mu < \nu$. If $\mathcal{S}'_p$ is MS-I on $[\mu, \nu]$ $\left|\mathcal{S}'_p\right|^\sigma$ is MT-convex on $[\mu, \nu]$, for some $\sigma > 1$ with $|\mathcal{S}'_p(\chi, \cdot)| \leq A$, then we have:*

$$\left|\frac{(\nu-\chi)\mathcal{S}_p(\nu,\cdot)+(\chi-\mu)\mathcal{S}_p(\mu,\cdot)}{(\nu-\mu)} - \frac{1}{(\nu-\mu)}\int_\mu^\nu \mathcal{S}_p(u,\cdot)du\right| \leq A\left(\frac{\xi\left(\frac{1}{2}\right)\xi\left(\sigma+\frac{1}{2}\right)}{2\xi(1+\sigma)}\right)^{\frac{1}{\sigma}} \frac{(\chi-\mu)^2+(\nu-\chi)^2}{(\nu-\mu)},$$

*for each $\chi \in [\mu, \nu]$.*

*Proof.* With the help of Hölder inequality on Lemma 2.1, we have:

$$\begin{aligned}
&\left|\frac{(\nu-\chi)\mathcal{S}_p(\nu,\cdot)+(\chi-\mu)\mathcal{S}_p(\mu,\cdot)}{(\nu-\mu)} - \frac{1}{(\nu-\mu)}\int_\mu^\nu \mathcal{S}_p(u,\cdot)du\right| \\
&\leq \frac{(\chi-\mu)^2}{(\nu-\mu)}\int_0^1 (1-\tau)\left|\mathcal{S}'_p\Big(\chi\tau+(1-\tau)\mu,\cdot\Big)\right|d\tau + \frac{(\nu-\chi)^2}{(\nu-\mu)}\int_0^1 (1-\tau)\left|\mathcal{S}'_p\Big(\chi+(1-\chi)\nu,\cdot\Big)\right|d\tau \\
&\leq \frac{(\chi-\mu)^2}{(\nu-\mu)}\left(\int_0^1 1d\tau\right)^{1-\frac{1}{\sigma}}\left(\int_0^1 (1-\tau)^\sigma\left|\mathcal{S}'_p\Big(\chi\tau+(1-\tau)\mu,\cdot\Big)\right|^\sigma d\tau\right)^{\frac{1}{\sigma}} \\
&\quad + \frac{(\nu-\chi)^2}{(\nu-\mu)}\left(\int_0^1 1d\tau\right)^{1-\frac{1}{\sigma}}\left(\int_0^1 (1-\tau)^\sigma\left|\mathcal{S}'_p\Big(\chi+(1-\chi)\nu\Big)\right|^\sigma d\tau\right)^{\frac{1}{\sigma}},
\end{aligned}$$

and since $|\mathcal{S}'_p|^\sigma$ is MT-convex stochastic process and $|\mathcal{S}'_p(\chi,\cdot)| \leq A$, then we have:

$$\begin{aligned}
&\int_0^1 (1-\tau)^\sigma\left|\mathcal{S}'_p\Big(\chi\tau+(1-\tau)\mu,\cdot\Big)\right|^\sigma d\tau \\
&\leq \int_0^1\left[\frac{(1-\tau)^\sigma\sqrt{\tau}}{2\sqrt{1-\tau}}|\mathcal{S}'_p(\chi,\cdot)|^\sigma + \frac{(1-\tau)^\sigma\sqrt{1-\tau}}{2\sqrt{\tau}}|\mathcal{S}'_p(a)|^\sigma\right]d\tau \\
&= \frac{1}{2}|\mathcal{S}'_p(\chi,\cdot)|^\sigma\int_0^1 \tau^{\frac{1}{2}}(1-\tau)^{\sigma-\frac{1}{2}}d\tau + \frac{1}{2}|\mathcal{S}'_p(\mu,\cdot)|^\sigma\int_0^1 \tau^{-\frac{1}{2}}(1-\tau)^{\sigma+\frac{1}{2}}d\tau \\
&\leq \frac{1}{2}A^\sigma\beta\left(\frac{3}{2},\sigma+\frac{1}{2}\right) + \frac{1}{2}A^\sigma\beta\left(\frac{1}{2},\sigma+\frac{3}{2}\right) = \frac{\xi\left(\frac{1}{2}\right)\xi\left(\sigma+\frac{1}{2}\right)}{2\xi(1+\sigma)}A^\sigma.
\end{aligned}$$

On the other hand we get similarly:

$$\begin{aligned}
&\int_0^1 (1-\tau)^\sigma|\mathcal{S}'_p(\chi\tau+(1-\tau)\nu,\cdot)|^\sigma d\tau \\
&\leq \int_0^1\left[\frac{(1-\tau)^\sigma\sqrt{\tau}}{2\sqrt{1-\tau}}|\mathcal{S}'_p(\chi,\cdot)|^\sigma + \frac{(1-\tau)^\sigma\sqrt{1-\tau}}{2\sqrt{\tau}}|\mathcal{S}'_p(\nu,\cdot)|^\sigma\right]d\tau \\
&\leq \frac{\xi\left(\frac{1}{2}\right)\xi\left(\sigma+\frac{1}{2}\right)}{2\xi(1+\sigma)}A^\sigma,
\end{aligned}$$

therefore we get the result. □

**Example 2.4.** *In Theorem 2.4, when $\chi = \dfrac{(\mu+\nu)}{2}$, the result becomes:*

$$\left|\frac{\mathcal{S}_p(\mu,\cdot)+\mathcal{S}_p(\nu,\cdot)}{2} - \frac{1}{(\nu-\mu)}\int_\mu^\nu \mathcal{S}_p(u,\cdot)\mathrm{d}u\right| \leq A\left(\frac{\xi\left(\frac{1}{2}\right)\xi\left(\sigma+\frac{1}{2}\right)}{\xi(1+\sigma)}\right)^{\frac{1}{\sigma}}\left(\frac{1}{2}\right)^{1+\frac{1}{\sigma}}(\nu-\mu).$$

## 3 Hermite-Hadamard Type Inequalities via Fractional Integral

In this section, we apply fractional integral identity as demonstrated in the following Lemma to extract some new Hermite-Hadamard type inequalities for MT-convex stochastic processes.

**Lemma 3.1.** *Let $\mathcal{S}_p : \mathcal{I} \times \mathcal{T} \to \mathbb{R}$ be a MS-D stochastic process on $I^\circ$, $a, b \in \mathcal{I}^\circ$ with $\mu < \nu$. If $\mathcal{S}'_p$ is MS-I on $[\mu, \nu]$, for all $\chi \in [\mu, \nu]$ and $\alpha > 0$, we get:*

$$\frac{(\chi-\mu)^\alpha \mathcal{S}_p(\mu,\cdot) + (\nu-\chi)^\alpha \mathcal{S}_p(\nu,\cdot)}{(\nu-\mu)} - \frac{\xi(\alpha+1)}{(\nu-\mu)}\Big[I^\alpha_{\chi^-}\mathcal{S}_p(\mu,\cdot) + I^\alpha_{\chi^+}\mathcal{S}_p(\nu,\cdot)\Big]$$
$$= \frac{(\chi-\mu)^{\alpha+1}}{(\nu-\mu)}\int_0^1 (\tau^\alpha - 1)\,\mathcal{S}'_p\Big(\chi\tau + (1-\tau)\mu,\cdot\Big)d\tau$$
$$+ \frac{(\nu-\chi)^{\alpha+1}}{(\nu-\mu)}\int_0^1 (1-\tau^\alpha)\,\mathcal{S}'_p\Big(\chi\tau + (1-\tau)\nu,\cdot\Big)d\tau,$$

*where $\xi(\alpha) = \int_0^\infty e^{-u}u^{\alpha-1}\,\mathrm{d}u$ is the Euler Gamma function.*

*Proof.* We have:

$$\int_0^1 (\tau^\alpha - 1)\,\mathcal{S}'_p\Big(\chi\tau + (1-\tau)\mu,\cdot\Big)d\tau$$
$$= \left[(\tau^\alpha - 1)\frac{\mathcal{S}_p\Big(\chi\tau+(1-\tau)\mu,\cdot\Big)}{(\chi-\mu)}\right]_0^1 - \int_0^1 \alpha\tau^{\alpha-1}\frac{\mathcal{S}_p\Big(\chi\tau+(1-\tau)\mu,\cdot\Big)}{(\chi-\mu)}d\tau$$
$$= \frac{\mathcal{S}_p(\mu,\cdot)}{(\chi-\mu)} - \frac{\alpha}{(\chi-\mu)}\int_\mu^\chi \left(\frac{u-a}{(\chi-\mu)}\right)^{\alpha-1}\frac{\mathcal{S}_p(u,\cdot)}{(\chi-\mu)}du$$
$$= \frac{\mathcal{S}_p(\mu,\cdot)}{(\chi-\mu)} - \frac{\alpha\xi(\alpha)}{(\chi-\mu)^{\alpha+1}}I^\alpha_{\chi^-}\mathcal{S}_p(\mu,\cdot),$$

similarly we get:

$$\int_0^1 (1-\tau^\alpha)\,\mathcal{S}'_p\Big(\chi\tau + (1-\tau)\nu,\cdot\Big)d\tau = \frac{\mathcal{S}_p(\nu,\cdot)}{\nu-\chi} - \frac{\alpha\xi(\alpha)}{(\nu-\chi)^{\alpha+1}}I^\alpha_{\chi^+}\mathcal{S}_p(\nu,\cdot).$$

By multiplying both sides of the identities above respectively by $\dfrac{(\chi-\mu)^{\alpha+1}}{(\nu-\mu)}$ and $\dfrac{(\nu-\chi)^{\alpha+1}}{(\nu-\mu)}$, then adding the results, we get the result. □

**Theorem 3.1.** *Let $\mathcal{S}_p : \mathcal{I} \times \mathcal{T} \to \mathbb{R}$ be a MS-D stochastic process on $I^\circ$, $a, b \in \mathcal{I}^\circ$ with $\mu < \nu$. If $\mathcal{S}'_p$ is MS-I on $[\mu, \nu]$ and $\left|\mathcal{S}'_p\right|$ is MT-convex on $[\mu, \nu]$, with $|\mathcal{S}'_p(\chi,\cdot)| \le A$, $\chi \in [\mu, \nu]$, then we have for $\alpha > 0$:*

$$\left|\frac{(\chi-\mu)^\alpha \mathcal{S}_p(\mu,\cdot) + (\nu-\chi)^\alpha \mathcal{S}_p(\nu,\cdot)}{(\nu-\mu)} - \frac{\xi(\alpha+1)}{(\nu-\mu)}\Big[I^\alpha_{\chi^-}\mathcal{S}_p(\mu,\cdot) + I^\alpha_{\chi^+}\mathcal{S}_p(\nu,\cdot)\Big]\right|$$
$$\le \frac{A\Big[(\chi-\mu)^{\alpha+1} + (\nu-\chi)^{\alpha+1}\Big]}{2(\nu-\mu)}\left[\pi - \frac{\xi\left(\alpha+\frac{1}{2}\right)\xi\left(\frac{1}{2}\right)}{\xi(\alpha+1)}\right],$$

*where $\xi$ is the Euler Gamma function.*

*Proof.* With the help of Lemma 3.1 and the MT-convexity of $|\mathcal{S}_p'|$, we have:

$$\begin{aligned}
&\left|\frac{(\chi-\mu)^\alpha \mathcal{S}_p(\mu,\cdot)+(\nu-\chi)^\alpha \mathcal{S}_p(\nu,\cdot)}{(\nu-\mu)}-\frac{\xi(\alpha+1)}{(\nu-\mu)}\left[I^\alpha_{\chi^-}\mathcal{S}_p(\mu,\cdot)+I^\alpha_{\chi^+}\mathcal{S}_p(\nu,\cdot)\right]\right|\\
&\quad\le \frac{(\chi-\mu)^{\alpha+1}}{(\nu-\mu)}\int_0^1 |\tau^\alpha-1|\left|\mathcal{S}_p'\Big(\chi\tau+(1-\tau)\mu,\cdot\Big)\right|d\tau\\
&\qquad+\frac{(\nu-\chi)^{\alpha+1}}{(\nu-\mu)}\int_0^1 |1-\tau^\alpha|\left|\mathcal{S}_p'\Big(\chi\tau+(1-\tau)\nu,\cdot\Big)\right|d\tau\\
&\quad\le \frac{A\Big[(\chi-\mu)^{\alpha+1}+(\nu-\chi)^{\alpha+1}\Big]}{2(\nu-\mu)}\int_0^1 (1-\tau^\alpha)\left(\tau^{1/2}(1-\tau)^{-1/2}+\tau^{-1/2}(1-\tau)^{1/2}\right)d\tau\\
&\quad= \frac{A\Big[(\chi-\mu)^{\alpha+1}+(\nu-\chi)^{\alpha+1}\Big]}{2(\nu-\mu)}\left[\pi-\frac{\xi\left(\alpha+\frac{1}{2}\right)\xi\left(\frac{1}{2}\right)}{\xi(\alpha+1)}\right],
\end{aligned}$$

where we use the Beta function of Euler type defined as follow:

$$\beta(x,y)=\int_0^1 \tau^{x-1}(1-\tau)^{y-1}d\tau=\frac{\xi(x)\xi(y)}{\xi(x+y)},\qquad \forall\quad x,y>0.$$

□

**Example 3.1.** *In Theorem 3.1, if* $\chi=\dfrac{(\mu+\nu)}{2}$, *then:*

$$\begin{aligned}
&\left|\frac{(\nu-\mu)^{\alpha-1}}{2^{\alpha-1}}\frac{\mathcal{S}_p(\mu,\cdot)+\mathcal{S}_p(\nu,\cdot)}{2}-\frac{\xi(\alpha+1)}{(\nu-\mu)}\left[I^\alpha_{\left(\frac{\mu+\nu}{2}\right)^-}\mathcal{S}_p(\mu,\cdot)+I^\alpha_{\left(\frac{\mu+\nu}{2}\right)^+}\mathcal{S}_p(\nu,\cdot)\right]\right|\\
&\quad\le \frac{A(\nu-\mu)^\alpha}{2^{\alpha+1}}\left[\pi-\frac{\xi\left(\alpha+\frac{1}{2}\right)\xi\left(\frac{1}{2}\right)}{\xi(\alpha+1)}\right].
\end{aligned}$$

**Example 3.2.** *For* $\alpha=1$ *we find the proven result of Theorem 2.1.*

**Theorem 3.2.** *Let* $\mathcal{S}_p:\mathcal{I}\times\mathcal{T}\to\mathbb{R}$ *be a MS-D stochastic process on* $I^\circ, a, b\in\mathcal{I}^\circ$ *with* $\mu<\nu$. *If* $\mathcal{S}_p'$ *is MS-I on* $[\mu,\nu]$ $\left|\mathcal{S}_p'\right|^\sigma$ *is MT-convex on* $[\mu,\nu]$, *for some* $\sigma>1$ *with* $|\mathcal{S}_p'(\chi,\cdot)|\le A, \chi\in[\mu,\nu]$, *and* $\alpha>0$, *then we have:*

$$\begin{aligned}
&\left|\frac{(\chi-\mu)^\alpha \mathcal{S}_p(\mu,\cdot)+(\nu-\chi)^\alpha \mathcal{S}_p(\nu,\cdot)}{(\nu-\mu)}-\frac{\xi(\alpha+1)}{(\nu-\mu)}\left[I^\alpha_{\chi^-}\mathcal{S}_p(\mu,\cdot)+I^\alpha_{\chi^+}\mathcal{S}_p(\nu,\cdot)\right]\right|\\
&\quad\le \frac{A\Big[(\chi-\mu)^{\alpha+1}+(\nu-\chi)^{\alpha+1}\Big]}{(\nu-\mu)}\left(\frac{\pi}{2}\right)^{\frac{1}{\sigma}}\left(\frac{\xi(1+\rho)\xi\left(\frac{1}{\alpha}\right)}{\alpha\xi\left(1+\rho+\frac{1}{\alpha}\right)}\right)^{\frac{1}{\rho}},
\end{aligned}$$

*where* $\xi$ *is the Euler Gamma function, and* $\dfrac{1}{\rho}+\dfrac{1}{\sigma}=1$.

*Proof.* From Lemma 3.1, and using the Hölder inequality, we have:

$$\begin{aligned}
&\left|\frac{(\chi-\mu)^{\alpha}\mathcal{S}_p(\mu,\cdot)+(\nu-\chi)^{\alpha}\mathcal{S}_p(\nu,\cdot)}{(\nu-\mu)}-\frac{\xi(\alpha+1)}{(\nu-\mu)}\left[I^{\alpha}_{\chi^-}\mathcal{S}_p(\mu,\cdot)+I^{\alpha}_{\chi^+}\mathcal{S}_p(\nu,\cdot)\right]\right|\\
&\leq \frac{(\chi-\mu)^{\alpha+1}}{(\nu-\mu)}\int_0^1|\tau^{\alpha}-1|\left|\mathcal{S}'_p\Big(\chi\tau+(1-\tau)\mu,\cdot\Big)\right|d\tau\\
&\quad+\frac{(\nu-\chi)^{\alpha+1}}{(\nu-\mu)}\int_0^1|1-\tau^{\alpha}|\left|\mathcal{S}'_p\Big(\chi\tau+(1-\tau)\nu,\cdot\Big)\right|d\tau\\
&\leq \frac{(\chi-\mu)^{\alpha+1}}{(\nu-\mu)}\left(\int_0^1(1-\tau^{\alpha})^{\rho}\,d\tau\right)^{\frac{1}{\rho}}\left(\int_0^1\left|\mathcal{S}'_p\Big(\chi\tau+(1-\tau)\mu,\cdot\Big)\right|^{\sigma}d\tau\right)^{\frac{1}{\sigma}}\\
&\quad+\frac{(\nu-\chi)^{\alpha+1}}{(\nu-\mu)}\left(\int_0^1(1-\tau^{\alpha})^{\rho}\,d\tau\right)^{\frac{1}{\rho}}\left(\int_0^1\left|\mathcal{S}'_p\Big(\chi\tau+(1-\tau)\nu,\cdot\Big)\right|^{\sigma}d\tau\right)^{\frac{1}{\sigma}},
\end{aligned}$$

and since $|\mathcal{S}'_p|^{\sigma}$ is MT-convex stochastic process and $|\mathcal{S}'_p(\chi,\cdot)|\leq A$, then we have:

$$\begin{aligned}
\int_0^1\left|\mathcal{S}'_p\Big(\chi\tau+(1-\tau)\mu,\cdot\Big)\right|^{\sigma}\mathrm{d}\tau &\leq \int_0^1\left[\frac{\sqrt{\tau}}{2\sqrt{1-\tau}}|\mathcal{S}'_p(\chi,\cdot)|^{\sigma}+\frac{\sqrt{1-\tau}}{2\sqrt{\tau}}|\mathcal{S}'_p(\mu,\cdot)|^{\sigma}\right]d\tau\\
&=\frac{\pi}{4}\Big[|\mathcal{S}'_p(\chi,\cdot)|^{\sigma}+|\mathcal{S}'_p(\mu,\cdot)|^{\sigma}\Big]\leq\frac{\pi}{2}A^{\sigma},\\
\int_0^1\left|\mathcal{S}'_p\Big(\chi\tau+(1-\tau)\nu,\cdot\Big)\right|^{\sigma}\mathrm{d}\tau &\leq \int_0^1\left[\frac{\sqrt{\tau}}{2\sqrt{1-\tau}}|\mathcal{S}'_p(\chi,\cdot)|^{\sigma}+\frac{\sqrt{1-\tau}}{2\sqrt{\tau}}|\mathcal{S}'_p(\nu,\cdot)|^{\sigma}\right]d\tau\\
&=\frac{\pi}{4}\Big[|\mathcal{S}'_p(\chi,\cdot)|^{\sigma}+|\mathcal{S}'_p(\nu,\cdot)|^{\sigma}\Big]\leq\frac{\pi}{2}A^{\sigma},
\end{aligned}$$

and,

$$\int_0^1(1-\tau^{\alpha})^{\rho}\,d\tau=\frac{1}{\alpha}\int_0^1(1-s)^{\rho}s^{\frac{1}{\alpha}-1}ds=\frac{\xi(1+\rho)\xi\left(\frac{1}{\alpha}\right)}{\alpha\xi\left(1+\rho+\frac{1}{\alpha}\right)}.$$

Therefore we have:

$$\begin{aligned}
&\left|\frac{(\chi-\mu)^{\alpha}\mathcal{S}_p(\mu,\cdot)+(\nu-\chi)^{\alpha}\mathcal{S}_p(\nu,\cdot)}{(\nu-\mu)}-\frac{\xi(\alpha+1)}{(\nu-\mu)}\left[I^{\alpha}_{\chi^-}\mathcal{S}_p(\mu,\cdot)+I^{\alpha}_{\chi^+}\mathcal{S}_p(\nu,\cdot)\right]\right|\\
&\leq\frac{(\chi-\mu)^{\alpha+1}}{(\nu-\mu)}\left(\frac{\xi(1+\rho)\xi\left(\frac{1}{\alpha}\right)}{\alpha\xi\left(1+\rho+\frac{1}{\alpha}\right)}\right)^{\frac{1}{\rho}}\left(\frac{\pi A^{\sigma}}{2}\right)^{\frac{1}{\sigma}}+\frac{(\nu-\chi)^{\alpha+1}}{(\nu-\mu)}\left(\frac{\xi(1+\rho)\xi\left(\frac{1}{\alpha}\right)}{\alpha\xi\left(1+\rho+\frac{1}{\alpha}\right)}\right)^{\frac{1}{\rho}}\left(\frac{\pi A^{\sigma}}{2}\right)^{\frac{1}{\sigma}},
\end{aligned}$$

which completes the proof. □

**Example 3.3.** *In Theorem 3.2, by replacing* $\chi$ *with* $\dfrac{(\mu+\nu)}{2}$*, we obtain:*

$$\begin{aligned}
&\left|\frac{(\nu-\mu)^{\alpha-1}}{2^{\alpha-1}}\frac{\mathcal{S}_p(\mu,\cdot)+\mathcal{S}_p(\nu,\cdot)}{2}-\frac{\xi(\alpha+1)}{(\nu-\mu)}\left[I^{\alpha}_{\left(\frac{\mu+\nu}{2}\right)^-}\mathcal{S}_p(\mu,\cdot)+I^{\alpha}_{\left(\frac{\mu+\nu}{2}\right)^+}\mathcal{S}_p(\nu,\cdot)\right]\right|\\
&\leq\frac{A(\nu-\mu)^{\alpha}}{2^{\alpha}}\left(\frac{\pi}{2}\right)^{\frac{1}{\sigma}}\left(\frac{\xi(1+\rho)\xi\left(\frac{1}{\alpha}\right)}{\alpha\xi\left(1+\rho+\frac{1}{\alpha}\right)}\right)^{\frac{1}{\rho}}.
\end{aligned}$$

**Example 3.4.** *For* $\alpha=1$*, we get the result of Theorem 2.2.*

**Theorem 3.3.** *Let $\mathcal{S}_p : \mathcal{I} \times \mathcal{T} \to \mathbb{R}$ be a MS-D stochastic process on $I^\circ, a, b \in \mathcal{I}^\circ$ with $\mu < \nu$. If $\mathcal{S}'_p$ is MS-I on $[\mu, \nu]$ $|\mathcal{S}'_p|^\sigma$ is MT-convex on $[\mu, \nu]$, for some $\sigma > 1$ with $|\mathcal{S}'_p(\chi, \cdot)| \leq A, \chi \in [\mu, \nu]$, and $\alpha > 0$, then we have:*

$$\left| \frac{(\chi-\mu)^\alpha \mathcal{S}_p(\mu,\cdot) + (\nu-\chi)^\alpha \mathcal{S}_p(\nu,\cdot)}{(\nu-\mu)} - \frac{\xi(\alpha+1)}{(\nu-\mu)} \left[ I^\alpha_{\chi^-} \mathcal{S}_p(\mu,\cdot) + I^\alpha_{\chi^+} \mathcal{S}_p(\nu,\cdot) \right] \right|$$
$$\leq \frac{A\left[(\chi-\mu)^{\alpha+1} + (\nu-\chi)^{\alpha+1}\right]}{(\nu-\mu)} \left( \frac{\alpha}{\alpha+1} \right)^{1-\frac{1}{\sigma}} \left[ \frac{\pi}{2} - \frac{\xi\left(\alpha+\frac{1}{2}\right)\xi\left(\frac{1}{2}\right)}{2\xi(\alpha+1)} \right]^{\frac{1}{\sigma}},$$

*with $\xi$ being the Euler Gamma function.*

*Proof.* From Lemma 3.1, and by applying the Hölder inequality, we have:

$$\left| \frac{(\chi-\mu)^\alpha \mathcal{S}_p(\mu,\cdot) + (\nu-\chi)^\alpha \mathcal{S}_p(\nu,\cdot)}{(\nu-\mu)} - \frac{\xi(\alpha+1)}{(\nu-\mu)} \left[ I^\alpha_{\chi^-} \mathcal{S}_p(\mu,\cdot) + I^\alpha_{\chi^+} \mathcal{S}_p(\nu,\cdot) \right] \right|$$
$$\leq \frac{(\chi-\mu)^{\alpha+1}}{(\nu-\mu)} \int_0^1 |\tau^\alpha - 1| \left| \mathcal{S}'_p\Big(\chi\tau + (1-\tau)\mu, \cdot\Big) \right| d\tau$$
$$+ \frac{(\nu-\chi)^{\alpha+1}}{(\nu-\mu)} \int_0^1 |1-\tau^\alpha| \left| \mathcal{S}'_p\Big(\chi\tau + (1-\tau)\nu, \cdot\Big) \right| d\tau$$
$$\leq \frac{(\chi-\mu)^{\alpha+1}}{(\nu-\mu)} \left( \int_0^1 (1-\tau^\alpha)\, d\tau \right)^{1-\frac{1}{\sigma}} \left( \int_0^1 (1-\tau^\alpha) \left| \mathcal{S}'_p\Big(\chi\tau + (1-\tau)\mu, \cdot\Big) \right|^\sigma d\tau \right)^{\frac{1}{\sigma}}$$
$$+ \frac{(\nu-\chi)^{\alpha+1}}{(\nu-\mu)} \left( \int_0^1 (1-\tau^\alpha)\, d\tau \right)^{1-\frac{1}{\sigma}} \left( \int_0^1 (1-\tau^\alpha) \left| \mathcal{S}'_p\Big(\chi\tau + (1-\tau)\nu, \cdot\Big) \right|^\sigma d\tau \right)^{\frac{1}{\sigma}}.$$

The MT-convexity of $|\mathcal{S}'_p|^\sigma$ with $|\mathcal{S}'_p(\chi, \cdot)| \leq A$, give us:

$$\int_0^1 (1-\tau^\alpha) \left| \mathcal{S}'_p\Big(\chi\tau + (1-\tau)\mu, \cdot\Big) \right|^\sigma d\tau \leq \frac{A^\sigma}{2} \left( \pi - \frac{\xi\left(\alpha+\frac{1}{2}\right)\xi\left(\frac{1}{2}\right)}{\xi(\alpha+1)} \right),$$

and,

$$\int_0^1 (1-\tau^\alpha) \left| \mathcal{S}'_p\Big(\chi\tau + (1-\tau)\nu, \cdot\Big) \right|^\sigma d\tau \leq \int_0^1 (1-\tau^\alpha) \left[ \frac{\sqrt{\tau}}{2\sqrt{1-\tau}} |\mathcal{S}'_p(\chi,\cdot)|^\sigma + \frac{\sqrt{1-\tau}}{2\sqrt{\tau}} |\mathcal{S}'_p(\nu,\cdot)|^\sigma \right] d\tau$$
$$\leq \frac{A^\sigma}{2} \left( \pi - \frac{\xi\left(\alpha+\frac{1}{2}\right)\xi\left(\frac{1}{2}\right)}{\xi(\alpha+1)} \right),$$

thus the result. □

**Example 3.5.** *For $\chi = \dfrac{(\mu+\nu)}{2}$ the inequality in Theorem 3.3 becomes:*

$$\left| \frac{(\nu-\mu)^{\alpha-1}}{2^{\alpha-1}} \frac{\mathcal{S}_p(\mu,\cdot) + \mathcal{S}_p(\nu,\cdot)}{2} - \frac{\xi(\alpha+1)}{(\nu-\mu)} \left[ I^\alpha_{\left(\frac{\mu+\nu}{2}\right)^-} \mathcal{S}_p(\mu,\cdot) + I^\alpha_{\left(\frac{\mu+\nu}{2}\right)^+} \mathcal{S}_p(\nu,\cdot) \right] \right|$$
$$\leq \frac{A(\nu-\mu)^\alpha}{2^\alpha} \left( \frac{\alpha}{\alpha+1} \right)^{1-\frac{1}{\sigma}} \left[ \frac{\pi}{2} - \frac{\xi\left(\alpha+\frac{1}{2}\right)\xi\left(\frac{1}{2}\right)}{2\xi(\alpha+1)} \right]^{\frac{1}{\sigma}}.$$

**Example 3.6.** *In the case of $\alpha = 1$, the result of Theorem 2.3 can be found.*

## 4 Conclusion

In this paper, we managed to obtained new estimates of the left-hand side of a Hermite-Hadamard type inequality for stochastic processes whose first derivatives absolute values are MT-convex by using the Riemann-Liouville fractional integral operator. The purpose of this paper is to extend the results of the Hermite-Hadamard inequality of convex stochastic processes, using Riemann-Liouville fractional integral operator, with the hope that it may inspire further studies in the field by being a starting point for other researchers to explore new insights into other types of convex stochastic processes.

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